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GLOBAL SOLVABILITY FOR DOUBLE-DIFFUSIVE CONVECTION SYSTEM BASED ON BRINKMAN–FORCHHEIMER EQUATION IN GENERAL DOMAINS

MITSUHARU ÔTANI* and SHUN UCHIDA†

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Abstract

In this paper, we are concerned with the solvability of the initial boundary value problem of a system which describes double-diffusive convection phenomena in some porous medium under general domains, especially unbounded domains. In previous works where the boundedness of the space domain is imposed, some global solvability results have been already derived. However, when we consider our problem in general domains, some compactness theorems are not available. Hence it becomes difficult to follow the same strategies as before. Nevertheless, we can assure the global existence of a unique solution via the contraction method. Moreover, it is revealed that the global solvability holds for higher space dimension and larger class of the initial data than those assumed in previous works.

1. Introduction

We consider the following double-diffusive convection system based upon Brinkman–Forchheimer equation.

$$
\text{(DCBF)}\begin{cases}
\partial_t \boldsymbol{u} = \nu\Delta\boldsymbol{u} - a\boldsymbol{u} - \nabla p + \boldsymbol{g}T + \boldsymbol{h}C + \boldsymbol{f}_1, & (x,t)\in\Omega\times[0,S],\\
\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2, & (x,t)\in\Omega\times[0,S],\\
\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3, & (x,t)\in\Omega\times[0,S],\\
\nabla\cdot\boldsymbol{u} = 0, & (x,t)\in\Omega\times[0,S],\\
\boldsymbol{u} = 0,\quad \dfrac{\partial T}{\partial n} = 0,\quad \dfrac{\partial C}{\partial n} = 0, & (x,t)\in\partial\Omega\times[0,S],\\
\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot),\quad T(\cdot,0) = T_0(\cdot),\quad C(\cdot,0) = C_0(\cdot), &
\end{cases}
$$

where $\Omega\subset\mathbb{R}^N$ is a general domain with some suitable conditions (provided later in Section 2.2) and $\partial\Omega$ is the boundary of Ω. The unit outward normal vector on $\partial\Omega$ is denoted by n and $\partial T/\partial n := \nabla T\cdot n$. Unknown functions of this system are $\boldsymbol{u}(x,t) =$

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$(u_1(x,t), u_2(x,t), \ldots, u_N(x,t))^t$, $T(x,t)$, $C(x,t)$ and $p(x,t)$ which represent the fluid velocity, the temperature of fluid, the concentration of solute and the pressure of fluid respectively. Positive constants ν, ρ and a are called the viscosity coefficient, Soret's coefficient and Darcy's coefficient respectively. Constant vectors $\boldsymbol{g} = (g_1, g_2, \ldots, g_N)^t$ and $\boldsymbol{h} = (h_1, h_2, \ldots, h_N)^t$ are derived from the gravity. The terms $\boldsymbol{f}_1(x,t) = (f_1^1(x,t), f_1^2(x,t), \ldots, f_1^N(x,t))^t$, $f_2(x,t)$ and $f_3(x,t)$ are given external forces.

Double-diffusive convection is a model of convection in the fluid reflecting some interactions between the temperature and the concentration of solute. When the distribution of the temperature is far from homogeneous, the behavior of the concentration of solute becomes more complicated than the simplified diffusion model. Double-diffusive convection phenomena can be described by the second equation and the third equation of (DCBF) which originate from results of the irreversible thermodynamics. The term $\rho\Delta T$ describes one of interactions between the temperature of fluid and the concentration of solute, the so-called Soret's effect. It is well known that double-diffusive convection is mainly characterized by this Soret's effect. Strictly speaking, the second equation also contains an interaction term $\rho'\Delta C$, which is called Dufour's effect. However, Dufour's effect is generally much smaller than Soret's effect, especially for the case where the fluid is a liquid. Therefore we here consider only Soret's effect.

A great number of researches in double-diffusive convection phenomena have been carried out (see, e.g., Brandt–Fernando [2] and Nield–Bejan [5]). Among them, the study of double-diffusive convection in porous media is one of subjects which attracted a lot of researchers, since the model of double-diffusive convection in porous media has a large area of application, for example, the behavior of polluted water in the soil. When we deal with these models, the void space of porous medium is assumed to be relatively large. In order to describe the behavior of the fluid velocity under these situations, it is appropriate to apply the so-called Brinkman–Forchheimer equation.

The first equation of (DCBF) is based on Brinkman–Forchheimer equation. Originally, Brinkman–Forchheimer equation has a convection term and another nonlinear term, and in each term of the equation, there appears another space-dependent function which stands for the rate of void space in the porous medium (which is called the porosity). However, we use a linearized Brinkman–Forchheimer equation in our system under some physical conditions, for example, homogeneity of the porous medium. Here $\boldsymbol{g}T$ and $\boldsymbol{h}C$ are the effects from the gravity.

For the case where the space domain Ω is bounded, there exist some results for global solvability. In [13], for instance, the initial boundary value problem of (DCBF) with homogeneous Dirichlet boundary condition is considered. They showed the existence of a unique global solution for the case where the initial data belongs to $H_0^1(\Omega)$ and the space dimension $N \leqslant 3$. Moreover, the time periodic problem of (DCBF) with homogeneous Dirichlet boundary condition is examined in [8], where the existence of time periodic solutions is showed for $N \leqslant 3$. In the case where homogeneous Neumann boundary condition is imposed, the solvability of the initial boundary value problem and the time periodic problem are solved in [9] with $N \leqslant 3$. In this way, some results for global solvability can be assured even for $N = 3$, in spite of the existence of convection terms $\boldsymbol{u} \cdot \nabla T$, $\boldsymbol{u} \cdot \nabla C$ which possess the nonlinearity quite similar to that appearing in the Navier–Stokes equations.

In previous works referred above, (DCBF) is reduced to some abstract problem in an appropriate Hilbert space and some abstract theories are applied, which are developed in [6] and [7], results for Cauchy problem and periodic problem of an abstract equation governed by subdifferential operators with non-monotone perturbations. When we apply these abstract results, Rellich–Kondrachov's compactness theorem plays an essential role. Therefore, it is difficult to follow the same strategy as before for unbounded domains.

The main purpose of this paper is to show that the global solvability result still holds true for (DCBF) with unbounded space domains. In order to carry out this, we rely on Banach's fixed point theorem. In Section 2, our main results are stated and some notations and function spaces are fixed. We give an outline of the proof for our main result in Section 3. In Section 4, we show the well-definedness of some mappings, to which the fixed point theorem will be applied. In Section 5, we assure that the composition of these mappings becomes a contraction mapping in some appropriate function space. Finally, we shall show that the time-local solution constructed in Sections 4 and 5 can be globally extended in Section 6.

2. Main result

2.1. Notation. In order to formulate our result, we use the following notation:

$\mathbb{C}_\sigma^\infty(\Omega) := \{\boldsymbol{u} = (u^1, u^2, \dots, u^N)^t;\ u^j \in C_0^\infty(\Omega),\ \forall j = 1, 2, \dots, N,\ \nabla \cdot \boldsymbol{u} = 0\}$,

$\mathbb{L}^2(\Omega) := (L^2(\Omega))^N, \quad \mathbb{H}^k(\Omega) := (H^k(\Omega))^N = (W^{k,2}(\Omega))^N$,

$\mathbb{L}^2_\sigma(\Omega)$: the closure of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{L}^2(\Omega)$-norm,

$\mathbb{H}^1_\sigma(\Omega)$: the closure of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{H}^1(\Omega)$-norm,

$\mathcal{P}_\Omega$: the orthogonal projection from $\mathbb{L}^2(\Omega)$ onto $\mathbb{L}^2_\sigma(\Omega)$,

$\mathcal{A} := -\mathcal{P}_\Omega \Delta$: the Stokes operator with domain $D(\mathcal{A}) = \mathbb{H}^2(\Omega) \cap \mathbb{H}^1_\sigma(\Omega)$.

Moreover, we define the following function spaces where solutions and external forces should belong.

$W_S := C([0, S]; \mathbb{H}^1_\sigma(\Omega)) \cap L^2(0, S; \mathbb{H}^2(\Omega))$,

$X_S := \{ f \in L^1(0, S; L^2(\Omega));\ \sqrt{t} f \in L^2(0, S; L^2(\Omega))\}$,

$Y_S := \{U \in C([0, S]; L^2(\Omega)) \cap L^2(0, S; H^1(\Omega));\ \sqrt{t}\Delta U,\ \sqrt{t}\partial_t U \in L^2(0, S; L^2(\Omega))\}$,

$$Z_S := \left\{ \begin{pmatrix} \boldsymbol{u} \\ T \\ C \end{pmatrix} \in C([0,\,S];\mathbb{L}^2_\sigma(\Omega)\times L^2(\Omega)\times L^2(\Omega));\ \begin{array}{l} \boldsymbol{u}\in W_S,\ T, C\in Y_S, \\ \partial_t \boldsymbol{u}\in L^2(0,\,S;\mathbb{L}^2_\sigma(\Omega)) \end{array} \right\}.$$

Here norms for W_S, X_S and Y_S are defined as follows respectively:

$$\begin{aligned}
\|\boldsymbol{u}\|_{W_S} &:= \sup_{0\leqslant t\leqslant S}\|\boldsymbol{u}(t)\|_{\mathbb{H}^1_\sigma(\Omega)} + \left(\int_0^S \|\boldsymbol{u}(t)\|^2_{\mathbb{H}^2(\Omega)}\,dt\right)^{1/2},\\
\|f\|_{X_S} &:= \|f\|_{L^1(0,S;L^2(\Omega))} + \left(\int_0^S t\|f(t)\|^2_{L^2(\Omega)}\,dt\right)^{1/2},\\
\|U\|_{Y_S} &:= \sup_{0\leqslant t\leqslant S}\|U(t)\|_{L^2(\Omega)} + \|\nabla U(t)\|_{L^2(0,S;L^2(\Omega))}\\
&\quad + \left(\int_0^S t\|\Delta U(t)\|^2_{L^2(\Omega)}\,dt\right)^{1/2} + \left(\int_0^S t\|\partial_t U(t)\|^2_{L^2(\Omega)}\,dt\right)^{1/2}.
\end{aligned}$$

2.2. Main results. Our main result can be stated as follows:

Theorem 2.1. *Let $N \leqslant 4$ and let the space domain Ω satisfy the following condition $\#\Omega$.*

$\#\Omega$: *The space domain Ω falls under any of the following*:
1. *Whole space $\mathbb{R}^N$.*
2. *A domain belonging to red uniformly C^2 (resp. C^3)-regular class for $N = 2, 3$ (resp. $N = 4$) (see, e.g., Amann* [1], *Browder* [4] *and Sohr* [10]).

Moreover, assume that the initial data satisfy $\boldsymbol{u}_0 \in \mathbb{H}^1_\sigma(\Omega)$, T_0, $C_0 \in L^2(\Omega)$ and the external forces satisfy $\boldsymbol{f}_1 \in L^2(0,S;\mathbb{L}^2(\Omega))$, f_2, $f_3 \in X_S$. Then, for each $S > 0$, (DCBF) *admits a unique solution $(\boldsymbol{u}, T, C)^t \in Z_S$.*

REMARK 1. (1) The following cases are sufficient to assure condition 2 of $\#\Omega$.
- A bounded domain with C^2 (or C^3)-class boundary.
- A exterior of some bounded domain with C^2 (or C^3)-class boundary.

(2) The assumption that Ω is uniform C^2-domain implies that we can apply Sobolev's embedding theorem and the elliptic estimates for the operators $-\Delta$ and $\mathcal{A}$. Therefore, throughout this paper, one can guarantee to use the following facts under condition $\#\Omega$, i.e., there exist some constants γ_s, γ_{el}, γ_{es} such that the following inequalities hold.

$$\begin{aligned}
&\|U\|_{L^{p^*}(\Omega)} \leqslant \gamma_s\|U\|_{W^{1,p}(\Omega)}, \quad \forall U\in W^{1,p}(\Omega)\ (1\leqslant p<N,\ 1/p^*=1/p-1/N),\\
&\|U\|_{H^2(\Omega)} \leqslant \gamma_{el}(\|U\|_{L^2(\Omega)}+\|\Delta U\|_{L^2(\Omega)}), \quad \forall U\in D(-\Delta)=\{U\in H^2(\Omega);\partial U/\partial n|_{\partial\Omega}=0\},\\
&\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)} \leqslant \gamma_{es}(\|\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}+\|\mathcal{A}\boldsymbol{u}\|_{\mathbb{L}^2(\Omega)}), \quad \forall \boldsymbol{u}\in D(\mathcal{A}),
\end{aligned}$$

where γ_{el}, γ_{es} depends only on Ω, N and γ_s is a embedding constant depending on Ω, N and p. Moreover, the assumption that Ω is uniform C^3-domain for $N = 4$ implies that the embedding $D(\mathcal{A}^2) \subset \mathbb{H}^3(\Omega)$ is valid (see Theorem 1.5.1, Section III in Sohr [10]), where $\mathcal{A}^2 = \mathcal{A} \circ \mathcal{A}$ and $D(\mathcal{A}^2)$ stands for its domain. Namely, under the condition #Ω, we obtain the embedding $C([0, S]; D(\mathcal{A}^2)) \subset L^\infty(0, S; \mathbb{L}^\infty(\Omega))$ for $N \leqslant 4$, which will be used in our proof of Lemma 4.1 later on.

(3) Even if the homogeneous Neumann boundary condition is replaced by the homogeneous Dirichlet boundary condition, we can derive almost the same result as Theorem 2.1.

(4) If $T_0, C_0 \in H^1(\Omega)$, we can show the existence of a unique solution with the same regularity as that derived in [13]. Indeed, we can carry out the same arguments as in proofs of Lemmas 4.1 and 4.2 given later without the weight $\sqrt{t}$ and with $T_0, C_0 \in L^2(\Omega)$ and $f_2, f_3 \in X_S$ replaced by $T_0, C_0 \in H^1(\Omega)$ and $f_2, f_3 \in L^2(0, S; L^2(\Omega))$ respectively, in order to construct a unique global solution belonging to the space Y'_S defined by

$$Y'_S := \{U \in C([0, S]; H^1(\Omega));\ \Delta U, \partial_t U \in L^2(0, S; L^2(\Omega))\},$$

with the norm

$$\|U\|_{Y'_S} := \sup_{0 \leqslant t \leqslant S} \|U(t)\|_{H^1(\Omega)} + \|\Delta U\|_{L^2(0,S;L^2(\Omega))} + \|\partial_t U\|_{L^2(0,S;L^2(\Omega))}.$$

Hence, we can obtain the following result.

Theorem 2.2. *Let $N \leqslant 4$ and let $\boldsymbol{f}_1 \in L^2(0, S; \mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0, S; L^2(\Omega))$. Moreover, let the initial data satisfy $\boldsymbol{u}_0 \in \mathbb{H}^1_\sigma(\Omega)$, $T_0, C_0 \in H^1(\Omega)$. Then, for each $S > 0$,* (DCBF) *admits a unique solution $(\boldsymbol{u}, T, C)^t$ satisfying the following regularities*:

$$\begin{cases} \boldsymbol{u} \in C([0, S]; \mathbb{H}^1_\sigma(\Omega)), \\ \mathcal{A}\boldsymbol{u}, \partial_t \boldsymbol{u} \in L^2(0, S; \mathbb{L}^2_\sigma(\Omega)), \\ T, C \in C([0, S]; H^1(\Omega)), \\ \Delta T, \Delta C, \partial_t T, \partial_t C \in L^2(0, S; L^2(\Omega)). \end{cases}$$

3. Outline of the proof for Theorem 2.1

Operating the projection $\mathcal{P}_\Omega$ to the first equation of (DCBF), we have the following equation:

$$\begin{cases} \partial_t \boldsymbol{u} + \nu \mathcal{A}\boldsymbol{u} + a\boldsymbol{u} = \mathcal{P}_\Omega \boldsymbol{g} T + \mathcal{P}_\Omega \boldsymbol{h} C + \mathcal{P}_\Omega \boldsymbol{f}_1, \\ \partial_t T - \Delta T + \boldsymbol{u} \cdot \nabla T = f_2, \\ \partial_t C - \Delta C + \boldsymbol{u} \cdot \nabla C = \rho \Delta T + f_3. \end{cases} \tag{1}$$

It is well known that the system of above equations is equivalent to our system (DCBF) (see, e.g., Temam [12]). Therefore, here and after, we consider system (1).

In this section, we give an outline of our proof. Our proof consists of four steps:

STEP 1: For each $\underline{\boldsymbol{u}}$ fixed in W_S, we consider the following problem in $Y_S \times Y_S$:

$$
(2) \qquad \begin{cases} \partial_t \underline{T} - \Delta \underline{T} + \underline{\boldsymbol{u}} \cdot \nabla \underline{T} = f_2, \\ \partial_t \underline{C} - \Delta \underline{C} + \underline{\boldsymbol{u}} \cdot \nabla \underline{C} = \rho \Delta \underline{T} + f_3, \\ \left.\dfrac{\partial \underline{T}}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial \underline{C}}{\partial n}\right|_{\partial\Omega} = 0, \\ \underline{T}(\,\cdot\,, 0) = T_0(\,\cdot\,), \quad \underline{C}(\,\cdot\,, 0) = C_0(\,\cdot\,), \end{cases}
$$

where $\boldsymbol{u}$ in the second and third equations of (1) are replaced by given $\underline{\boldsymbol{u}}$. Then, we define a mapping $\Phi_{T_0,C_0} \colon W_S \to Y_S \times Y_S$ by $\Phi_{T_0,C_0}(\underline{\boldsymbol{u}}) = (\underline{T}, \underline{C})^t$, where $(\underline{T}, \underline{C})^t$ denotes a unique global solution of (2).

STEP 2: By substituting T, C by the unique solution $\underline{T}$, $\underline{C}$ in the first equation of (1), we consider the following problem:

$$
(3) \qquad \begin{cases} \partial_t \bar{\boldsymbol{u}} + \nu \mathcal{A} \bar{\boldsymbol{u}} + a \bar{\boldsymbol{u}} = \mathcal{P}_\Omega \boldsymbol{g} \underline{T} + \mathcal{P}_\Omega \boldsymbol{h} \underline{C} + \mathcal{P}_\Omega \boldsymbol{f}_1, \\ \bar{\boldsymbol{u}}|_{\partial\Omega} = 0, \quad \bar{\boldsymbol{u}}(\,\cdot\,, 0) = \boldsymbol{u}_0(\,\cdot\,), \end{cases}
$$

and we show that (3) admits a unique global solution $\bar{\boldsymbol{u}}$ in W_S. Moreover, we define $\Psi_{\boldsymbol{u}_0} \colon Y_S \times Y_S \to W_S$ by $\Psi_{\boldsymbol{u}_0}((\underline{T}, \underline{C})^t) = \bar{\boldsymbol{u}}$.

STEP 3: We show that the mapping $\Psi_{\boldsymbol{u}_0} \circ \Phi_{T_0,C_0}$ becomes a contraction mapping in W_{S_0} for a sufficiently small $S_0 \in (0, S]$. Hence we can show that (1) has a unique local solution in Z_{S_0}.

STEP 4: Establishing appropriate a priori estimates, we assure that time-local solutions can be extended globally onto $[0, S]$ (the whole of the prescribed interval).

4. Construction of solutions for Steps 1 and 2

In what follows, we carry out the program given in the previous section. To begin with, we discuss the procedures Steps 1 and 2. Namely, we ensure the solvability of problems (2) and (3) in this section.

4.1. Well-definedness of $\boldsymbol{\Phi_{T_0,C_0}}$. First we show the unique solvability of (2).

Lemma 4.1. *Let $N \leqslant 4$ and assume that $T_0 \in L^2(\Omega)$, $\boldsymbol{u} \in W_S$ and $f_2 \in X_S$. Then the following problem* (4) *has a unique global solution T in Y_S.*

$$
(4) \qquad \begin{cases} \partial_t T - \Delta T + \boldsymbol{u} \cdot \nabla T = f_2 \quad \text{in} \quad \Omega \times [0, S], \\ \left.\dfrac{\partial T}{\partial n}\right|_{\partial\Omega} = 0, \quad T(\,\cdot\,, 0) = T_0(\,\cdot\,). \end{cases}
$$

Proof. We first consider the case where $\boldsymbol{u}$ possesses higher regularity. To this end, we recall that the following heat equation has a unique global solution $T \in Y_S$

for any $T_0 \in L^2(\Omega)$ and $f \in X_S$ (see, e.g., Brézis [3]).

$$(5)\qquad \begin{cases} \partial_t T - \Delta T = f \quad \text{in} \quad \Omega \times [0, S], \\ \left.\dfrac{\partial T}{\partial n}\right|_{\partial\Omega} = 0, \quad T(\,\cdot\,, 0) = T_0(\,\cdot\,). \end{cases}$$

Therefore, since $\boldsymbol{w} \cdot \nabla U \in L^2(0, S; L^2(\Omega)) \subset X_S$ for any $U \in L^2(0, S; H^1(\Omega))$ and $\boldsymbol{w} \in C([0, S]; D(\mathcal{A}^2))$ (where $\mathcal{A}^2 := \mathcal{A} \circ \mathcal{A}$), the following equation also has a unique global solution $T \in Y_S$.

$$(6)\qquad \begin{cases} \partial_t T - \Delta T + \boldsymbol{w} \cdot \nabla U = f_2 \quad \text{in} \quad \Omega \times [0, S], \\ \left.\dfrac{\partial T}{\partial n}\right|_{\partial\Omega} = 0, \quad T(\,\cdot\,, 0) = T_0(\,\cdot\,). \end{cases}$$

Then we can define a mapping $\Sigma_{T_0}^{\boldsymbol{w}}\colon L^2(0, S; H^1(\Omega)) \to Y_S \subset L^2(0, S; H^1(\Omega))$ by the correspondence $\Sigma_{T_0}^{\boldsymbol{w}}(U) = T$.

Let $\Sigma_{T_0}^{\boldsymbol{w}}(U_i) = T_i$ $(i = 1, 2)$ and $\delta U = U_1 - U_2$, $\delta T = T_1 - T_2$. Obviously, δU and δT satisfy the following problem:

$$(7)\qquad \begin{cases} \partial_t \delta T - \Delta \delta T + \boldsymbol{w} \cdot \nabla \delta U = 0 \quad \text{in} \quad \Omega \times [0, S], \\ \left.\dfrac{\partial \delta T}{\partial n}\right|_{\partial\Omega} = 0, \quad \delta T(\,\cdot\,, 0) = 0. \end{cases}$$

Multiplying (7) by δT, we have

$$(8)\qquad \frac{1}{2}\frac{d}{dt}\|\delta T\|_{L^2}^2 + \|\nabla \delta T\|_{L^2}^2 \leqslant \|\boldsymbol{w} \cdot \nabla \delta U\|_{L^2}\|\delta T\|_{L^2} \leqslant M\|\nabla \delta U\|_{L^2}\|\delta T\|_{L^2},$$

where M is the constant given by $M := \operatorname{ess\,sup}_{(x,t)\in\Omega\times[0,S]}|\boldsymbol{w}(x,t)|$ (we here remark that $C([0, S]; D(\mathcal{A}^2)) \subset L^\infty(0, S; \mathbb{L}^\infty(\Omega))$ for $N \leqslant 4$). Therefore, integrating (8) over $[0, t]$, we have

$$(9)\qquad \begin{aligned} \frac{1}{2}\|\delta T(t)\|_{L^2}^2 &\leqslant M \int_0^t \|\nabla \delta U\|_{L^2}\|\delta T\|_{L^2}\, ds \\ &\leqslant MS^{1/2} \sup_{0\leqslant t\leqslant S} \|\delta T(t)\|_{L^2}\|\nabla \delta U\|_{L^2(0,S;L^2(\Omega))}. \end{aligned}$$

Hence we obtain

$$(10)\qquad \sup_{0\leqslant t\leqslant S} \|\delta T(t)\|_{L^2} + \|\nabla \delta T\|_{L^2(0,S;L^2(\Omega))} \leqslant (2 + \sqrt{2})MS^{1/2}\|\nabla \delta U\|_{L^2(0,S;L^2(\Omega))}.$$

Thus, from (10), we can assure that $\Sigma_{T_0}^{\boldsymbol{w}}$ becomes a contraction mapping with a sufficiently small $S_0 \in (0, S]$, i.e., we can show that the following problem has a unique local solution $T \in Y_{S_0}$ for any smooth function $\boldsymbol{w} \in C([0, S]; D(\mathcal{A}^2))$.

$$
\text{(11)} \qquad \begin{cases} \partial_t T - \Delta T + \boldsymbol{w} \cdot \nabla T = f_2 \quad \text{in} \quad \Omega \times [0, S], \\ \left.\dfrac{\partial T}{\partial n}\right|_{\partial\Omega} = 0, \quad T(\,\cdot\,, 0) = T_0(\,\cdot\,). \end{cases}
$$

Moreover, it is easy to see that this local solution can be extended globally. Indeed, since

$$
\text{(12)} \qquad \int_\Omega T\boldsymbol{w} \cdot \nabla T\, dx = \frac{1}{2}\int_\Omega \boldsymbol{w} \cdot \nabla T^2\, dx = \frac{1}{2}\int_\Omega T^2 \nabla \cdot \boldsymbol{w}\, dx = 0,
$$

multiplying (11) by T, we can obtain a priori bound for $\|T(t)\|_{L^2}$.

We here remark that there exists a sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ satisfying $\boldsymbol{u}_n \in C([0,S]; D(\mathcal{A}^2))$ and $\boldsymbol{u}_n \to \boldsymbol{u}$ in $C([0,S]; \mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,S; \mathbb{H}^2(\Omega))$ for each $\boldsymbol{u} \in W_S$ (for a typical example, see Remark 2 given later). Then, for each $n \in \mathbb{N}$, the following problem has a unique global solution $T_n \in Y_S$.

$$
\text{(13)} \qquad \begin{cases} \partial_t T_n - \Delta T_n + \boldsymbol{u}_n \cdot \nabla T_n = f_2 \quad \text{in} \quad [0, S] \times \Omega, \\ \left.\dfrac{\partial T_n}{\partial n}\right|_{\partial\Omega} = 0, \quad T_n(\,\cdot\,, 0) = T_0(\,\cdot\,) \in L^2(\Omega). \end{cases}
$$

In order to show the existence of a global solution of (4) for $\boldsymbol{u} \in W_S$, we here discuss the convergence of approximate solutions T_n. To this end, we begin with establishing some a priori estimates for T_n. Multiplying (13) by T_n, we have

$$
\text{(14)} \qquad \frac{1}{2}\frac{d}{dt}\|T_n\|^2_{L^2} + \|\nabla T_n\|^2_{L^2} \leqslant \|f_2\|_{L^2}\|T_n\|_{L^2},
$$

since $\int_\Omega T_n \boldsymbol{u}_n \cdot \nabla T_n\, dx = 0$. Hence, we obtain

$$
\text{(15)} \qquad \sup_{0\leqslant t\leqslant S} \|T_n(t)\|_{L^2} + \|\nabla T_n\|_{L^2(0,S;L^2(\Omega))} \leqslant \gamma_1,
$$

where γ_1 denotes a general constant independent of n. Next, multiplying (13) by $-t\Delta T_n$, we have

$$
\text{(16)} \qquad \frac{1}{2}\frac{d}{dt}t\|\nabla T_n\|^2_{L^2} + \frac{t}{4}\|\Delta T_n\|^2_{L^2} \leqslant t\|\boldsymbol{u}_n \cdot \nabla T_n\|^2_{L^2} + \frac{t}{2}\|f_2\|^2_{L^2} + \frac{1}{2}\|\nabla T_n\|^2_{L^2}.
$$

Here, applying Hölder's inequality, Sobolev's inequality, Riesz–Thorin's interpolation theorem and the elliptic estimate, we can derive the following estimate:

$$
\begin{aligned}
\|\boldsymbol{u}_n\cdot\nabla T_n\|_{L^2}^2 &\leqslant \|\boldsymbol{u}_n\|_{\mathbb{L}^8}^2\|\nabla T_n\|_{L^{8/3}}^2 \leqslant \gamma_1\|\boldsymbol{u}_n\|_{\mathbb{W}^{1,8/3}}^2\|\nabla T_n\|_{L^{8/3}}^2\\
&\leqslant \gamma_1\|\boldsymbol{u}_n\|_{\mathbb{H}^1}\|\boldsymbol{u}_n\|_{\mathbb{W}^{1,4}}\|\nabla T_n\|_{L^2}\|\nabla T_n\|_{L^4}\\
&\leqslant \gamma_1\|\boldsymbol{u}_n\|_{\mathbb{H}^1}\|\boldsymbol{u}_n\|_{\mathbb{H}^2}\|\nabla T_n\|_{L^2}\|T_n\|_{H^2}\\
&\leqslant \gamma_1\|\boldsymbol{u}_n\|_{\mathbb{H}^1}\|\boldsymbol{u}_n\|_{\mathbb{H}^2}\|\nabla T_n\|_{L^2}(\|\Delta T_n\|_{L^2}+\|T_n\|_{L^2}).
\end{aligned}
\tag{17}
$$

Then, by substituting (17) into (16), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}t\|\nabla T_n\|_{L^2}^2+\frac{t}{8}\|\Delta T_n\|_{L^2}^2\\
&\leqslant \gamma_1 t\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\|\nabla T_n\|_{L^2}^2+\frac{t}{2}\|f_2\|_{L^2}^2+\frac{t}{2}\|T_n\|_{L^2}^2+\frac{1}{2}\|\nabla T_n\|_{L^2}^2.
\end{aligned}
\tag{18}
$$

Moreover, applying Gronwall's inequality to (18), we obtain

$$
t\|\nabla T_n(t)\|_{L^2}^2 \leqslant \int_0^S (s\|f_2\|_{L^2}^2+s\|T_n\|_{L^2}^2+\|\nabla T_n\|_{L^2}^2)\,ds\,\exp\left(2\gamma_1\int_0^S\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\,ds\right).
\tag{19}
$$

Therefore, we obtain $\sup_{0\leqslant t\leqslant S} t\|\nabla T_n(t)\|_{L^2}^2 \leqslant \gamma_1$ with some constant γ_1 independent of n and $\int_0^S t\|\Delta T_n\|_{L^2}^2\,dt \leqslant \gamma_1$ by integrating (18) over $[0, S]$. Similarly, multiplying (13) by $t\partial_t T_n$, we have

$$
\begin{aligned}
\frac{t}{2}\|\partial_t T_n\|_{L^2}^2+\frac{d}{dt}\frac{t}{2}\|\nabla T_n\|_{L^2}^2 &\leqslant \gamma_1 t\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\|\nabla T_n\|_{L^2}^2+t\|f_2\|_{L^2}^2\\
&\quad+t(\|\Delta T_n\|_{L^2}+\|T_n\|_{L^2})^2+\frac{1}{2}\|\nabla T_n\|_{L^2}^2,
\end{aligned}
$$

whence follows $\int_0^S t\|\partial_t T_n\|_{L^2}^2\,dt \leqslant \gamma_1$.

By using these estimates, we show that $\{T_n\}$ becomes a Cauchy sequence in Y_S. Let $\delta\boldsymbol{u}=\boldsymbol{u}_m-\boldsymbol{u}_n$, $\delta T = T_m - T_n$. Then $\delta\boldsymbol{u}$ and δT satisfy the following equation:

$$
\begin{cases}
\partial_t\delta T-\Delta\delta T+\delta\boldsymbol{u}\cdot\nabla T_m+\boldsymbol{u}_n\cdot\nabla\delta T=0 \quad \text{in} \quad \Omega\times[0,S],\\
\left.\dfrac{\partial\delta T}{\partial n}\right|_{\partial\Omega}=0, \quad \delta T(\,\cdot\,,0)=0.
\end{cases}
\tag{20}
$$

Multiplying (20) by δT, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\delta T\|_{L^2}^2+\|\nabla\delta T\|_{L^2}^2\\
&=-\int_\Omega \delta T\,\delta\boldsymbol{u}\cdot\nabla T_m\,dx=\int_\Omega T_m\delta\boldsymbol{u}\cdot\nabla\delta T\,dx\\
&\leqslant \frac{1}{2}\|\nabla\delta T\|_{L^2}^2+\frac{1}{2}\|T_m\delta\boldsymbol{u}\|_{\mathbb{L}^2}^2 \leqslant \frac{1}{2}\|\nabla\delta T\|_{L^2}^2+\gamma_1\|T_m\|_{H^1}^2\|\delta\boldsymbol{u}\|_{\mathbb{H}^1}^2.
\end{aligned}
\tag{21}
$$

Therefore, integrating (21) and using (15), we can derive the following estimate of δT:

$$\sup_{0\leqslant t\leqslant S}\|\delta T(t)\|_{L^2}^2+\int_0^S\|\nabla\delta T\|_{L^2}^2\,ds\leqslant\gamma_1\sup_{0\leqslant t\leqslant S}\|\delta\boldsymbol{u}\|_{\mathbb{H}^1}^2. \tag{22}$$

Next, multiplying (20) by $-t\Delta\delta T$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}t\|\nabla\delta T\|_{L^2}^2+\frac{t}{8}\|\Delta\delta T\|_{L^2}^2\\&\leqslant\gamma_1\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2t\|\nabla\delta T\|_{L^2}^2+\frac{t}{2}\|\delta T\|_{L^2}^2+\frac{1}{2}\|\nabla\delta T\|_{L^2}^2\\&\quad+\gamma_1\|\delta\boldsymbol{u}\|_{\mathbb{H}^1}\|\delta\boldsymbol{u}\|_{\mathbb{H}^2}\sqrt{t}\|\nabla T_m\|_{L^2}\sqrt{t}(\|\Delta T_m\|_{L^2}+\|T_m\|_{L^2}),\end{aligned}\tag{23}$$

where we used the following estimates of convection terms derived by procedures similar to that in (17):

$$\begin{aligned}&\frac{t}{2}\|\delta\boldsymbol{u}\cdot\nabla T_m\|_{L^2}^2\leqslant\gamma_1\|\delta\boldsymbol{u}\|_{\mathbb{H}^1}\|\delta\boldsymbol{u}\|_{\mathbb{H}^2}\sqrt{t}\|\nabla T_m\|_{L^2}\sqrt{t}(\|\Delta T_m\|_{L^2}+\|T_m\|_{L^2}),\\&t\|\boldsymbol{u}_n\cdot\nabla\delta T\|_{L^2}^2\leqslant\gamma_1t\|\boldsymbol{u}_n\|_{\mathbb{H}^1}\|\boldsymbol{u}_n\|_{\mathbb{H}^2}\|\nabla\delta T\|_{L^2}(\|\Delta\delta T\|_{L^2}+\|\delta T\|_{L^2}).\end{aligned}\tag{24}$$

Since $\sup_{0\leqslant t\leqslant S}\|\boldsymbol{u}_n\|_{\mathbb{H}^1}$, $\int_0^S\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\,ds$ and $\sup_{0\leqslant t\leqslant S}\sqrt{t}\|\nabla T_m\|_{L^2}$ are uniformly bounded, we can derive the following inequality by applying Gronwall's inequality to (23):

$$\begin{aligned}&t\|\nabla\delta T(t)\|_{L^2}^2\\&\leqslant\gamma_1\int_0^S\{\|\delta\boldsymbol{u}\|_{\mathbb{H}^1}\|\delta\boldsymbol{u}\|_{\mathbb{H}^2}\sqrt{s}(\|\Delta T_m\|_{L^2}+\|T_m\|_{L^2})+s\|\delta T\|_{L^2}^2+\|\nabla\delta T\|_{L^2}^2\}\,ds.\end{aligned}\tag{25}$$

Moreover, by using the uniform boundedness of $\int_0^S t\|\Delta T_m\|_{L^2}^2\,dt$ and (22), we can obtain

$$\sup_{0\leqslant t\leqslant S}t\|\nabla\delta T(t)\|_{L^2}^2\leqslant\gamma_1\left(\sup_{0\leqslant t\leqslant S}\|\delta\boldsymbol{u}(t)\|_{\mathbb{H}^1}^2+\|\delta\boldsymbol{u}\|_{L^2(0,S;\mathbb{H}^2(\Omega))}^2\right). \tag{26}$$

Hence again by (23), we have

$$\int_0^S t\|\Delta\delta T\|_{L^2}^2\,dt\leqslant\gamma_1\left(\sup_{0\leqslant t\leqslant S}\|\delta\boldsymbol{u}(t)\|_{\mathbb{H}^1}^2+\|\delta\boldsymbol{u}\|_{L^2(0,S;\mathbb{H}^2(\Omega))}^2\right). \tag{27}$$

Similarly, multiplying (20) by $t\partial_t\delta T$, we can derive

$$\int_0^S t\|\partial_t\delta T\|_{L^2}^2\,dt\leqslant\gamma_1\left(\sup_{0\leqslant t\leqslant S}\|\delta\boldsymbol{u}(t)\|_{\mathbb{H}^1}^2+\|\delta\boldsymbol{u}\|_{L^2(0,S;\mathbb{H}^2(\Omega))}^2\right). \tag{28}$$

Therefore, from (22), (27) and (28), we can assure that $\{T_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y_S, hence (4) has a unique global solution in Y_S. □

REMARK 2. We can construct the approximation sequence $\{\boldsymbol{u}_n\}$ of $\boldsymbol{u} \in W_S$ used in (13) by the following procedure. Let J_n be the resolvent of $\mathcal{A}$, i.e., $J_n := (I + (1/n)\mathcal{A})^{-1}$ (we note that $J_n \boldsymbol{w} \in D(\mathcal{A}^2)$ is valid for any $\boldsymbol{w} \in D(\mathcal{A})$ due to the definition of J_n and due to the elliptic regularity for the Stokes operator $\mathcal{A}$). Moreover, let $\tilde{\boldsymbol{v}}$ be the extension of $\boldsymbol{v} \in C([0, S]; \mathbb{L}^2(\Omega))$ defined by

$$\tilde{\boldsymbol{v}}(t) = \begin{cases} \boldsymbol{v}(t), & t \in [0, S], \\ \boldsymbol{v}(-t), & t \in [-S, 0], \\ \boldsymbol{v}(2S - t), & t \in [S, 2S], \\ 0, & t \in \mathbb{R} \setminus [-S, 2S], \end{cases}$$

(we can adopt other extensions of $\boldsymbol{v} \in C([0, S]; \mathbb{L}^2(\Omega))$ instead of $\tilde{\boldsymbol{v}}$ if their extensions also assure the arguments given below). Then it can be shown that $\boldsymbol{u}_n := (\rho_{1/n} * \widetilde{J_n \boldsymbol{u}})|_{[0,S]}$ satisfies $\boldsymbol{u}_n \in C([0, S]; D(\mathcal{A}^2))$ and $\boldsymbol{u}_n \to \boldsymbol{u}$ in W_S as $n \to \infty$, where $\rho_{1/n}$ is the Friedrichs mollifier. Indeed, we get

$$\begin{aligned} \boldsymbol{u}(0) - \boldsymbol{u}_n(0) &= \boldsymbol{u}(0) - J_n \boldsymbol{u}(0) + J_n \boldsymbol{u}(0) - \boldsymbol{u}_n(0) \\ &= \boldsymbol{u}(0) - J_n \boldsymbol{u}(0) + \int_{-1/n}^{1/n} \rho_{1/n}(s)(J_n \boldsymbol{u}(0) - \widetilde{J_n \boldsymbol{u}}(-s))\, ds \\ &= \boldsymbol{u}(0) - J_n \boldsymbol{u}(0) + \int_{0}^{1/n} \rho_{1/n}(s)(J_n \boldsymbol{u}(0) - J_n \boldsymbol{u}(s))\, ds \\ &\quad + \int_{-1/n}^{0} \rho_{1/n}(s)(J_n \boldsymbol{u}(0) - J_n \boldsymbol{u}(-s))\, ds. \end{aligned}$$

Since J_n is a contraction mapping on $\mathbb{L}^2_\sigma(\Omega)$ and $\boldsymbol{u}$ belongs to $C([0,S];\mathbb{L}^2_\sigma(\Omega))$, we can see that $\boldsymbol{u}_n(0) \to \boldsymbol{u}(0)$ in $\mathbb{L}^2_\sigma(\Omega)$ as $n \to \infty$. Likewise, we can obtain the convergence $\boldsymbol{u}_n(t) \to \boldsymbol{u}(t)$ for any $t \in [0, S]$. Moreover, from the uniform continuity of $\boldsymbol{u}$ on $[0, S]$, we can assure that $\boldsymbol{u}_n$ converges to $\boldsymbol{u}$ in $C([0, S]; \mathbb{L}^2_\sigma(\Omega))$ (uniform convergence).

Let $\mathcal{A}^{1/2}$ denote the fractional power of the Stokes operator $\mathcal{A}$ of order $1/2$. Then $\mathcal{A}^{1/2}$ satisfies $D(\mathcal{A}^{1/2}) = \mathbb{H}^1_\sigma(\Omega)$ and $\|\mathcal{A}^{1/2}\boldsymbol{w}\|_{\mathbb{L}^2_\sigma} = \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}$ for any $\boldsymbol{w} \in D(\mathcal{A}^{1/2})$ (see, e.g., Sohr [10] and Tanabe [11]). By the linearity and closedness of $\mathcal{A}^{1/2}$ and by the fact that $\boldsymbol{u} \in C([0, S]; \mathbb{H}^1_\sigma(\Omega))$, we have

$$\begin{aligned} \mathcal{A}^{1/2}\boldsymbol{u}(0) - \mathcal{A}^{1/2}\boldsymbol{u}_n(0) &= \mathcal{A}^{1/2}\boldsymbol{u}(0) - J_n \mathcal{A}^{1/2}\boldsymbol{u}(0) \\ &\quad + \int_0^{1/n} \rho_{1/n}(s)(J_n \mathcal{A}^{1/2}\boldsymbol{u}(0) - J_n \mathcal{A}^{1/2}\boldsymbol{u}(s))\, ds \\ &\quad + \int_{-1/n}^{0} \rho_{1/n}(s)(J_n \mathcal{A}^{1/2}\boldsymbol{u}(0) - J_n \mathcal{A}^{1/2}\boldsymbol{u}(-s))\, ds. \end{aligned}$$

Therefore, we obtain $\mathcal{A}^{1/2}\boldsymbol{u}_n(0) \to \mathcal{A}^{1/2}\boldsymbol{u}(0)$ in $\mathbb{L}^2_\sigma(\Omega)$ since $\boldsymbol{u} \in C([0, S]; \mathbb{H}^1_\sigma(\Omega))$. By the same argument as above, we can assure that $\mathcal{A}^{1/2}\boldsymbol{u}_n \to \mathcal{A}^{1/2}\boldsymbol{u}$ in $C([0, S]; \mathbb{L}^2_\sigma(\Omega))$.

Hence $\boldsymbol{u}_n$ converges to $\boldsymbol{u}$ in $C([0, S]; \mathbb{H}^1_\sigma(\Omega))$.

Since $\widetilde{\mathcal{A}\boldsymbol{u}} \in L^2(\mathbb{R}^1; \mathbb{L}^2_\sigma(\Omega))$, it is easy to see that $\rho_{1/n} * \widetilde{\mathcal{A}\boldsymbol{u}}|_{[0,S]} \to \mathcal{A}\boldsymbol{u}$ in $L^2(0, S; \mathbb{L}^2_\sigma(\Omega))$ as $n \to \infty$. We can also derive that

$$\|\rho_{1/n} * \widetilde{\mathcal{A}\boldsymbol{u}}(t) - \rho_{1/n} * \widetilde{J_n\mathcal{A}\boldsymbol{u}}(t)\|_{\mathbb{L}^2_\sigma} \leqslant \rho_{1/n} * \|\widetilde{\mathcal{A}\boldsymbol{u}} - \widetilde{J_n\mathcal{A}\boldsymbol{u}}\|_{\mathbb{L}^2_\sigma}(t).$$

Therefore, by using Young's inequality, we have

$$\begin{aligned}\|\rho_{1/n} * \widetilde{\mathcal{A}\boldsymbol{u}} - \rho_{1/n} * \widetilde{J_n\mathcal{A}\boldsymbol{u}}\|_{L^2(\mathbb{R}^1;\mathbb{L}^2_\sigma(\Omega))} &\leqslant \|\rho_{1/n}\|_{L^1(\mathbb{R}^1)}\|\widetilde{\mathcal{A}\boldsymbol{u}} - \widetilde{J_n\mathcal{A}\boldsymbol{u}}\|_{L^2(\mathbb{R}^1;\mathbb{L}^2_\sigma(\Omega))}\\ &\leqslant 3\|\mathcal{A}\boldsymbol{u} - J_n\mathcal{A}\boldsymbol{u}\|_{L^2(0,S;\mathbb{L}^2_\sigma(\Omega))}.\end{aligned}$$

Since the right hand side converges to zero by virtue of Lebesgue's dominated convergence theorem, we can assure that $\rho_{1/n} * \widetilde{J_n\mathcal{A}\boldsymbol{u}}$ strongly converges to $\mathcal{A}\boldsymbol{u}$ in $L^2(0, S; \mathbb{L}^2_\sigma(\Omega))$.

Next, we consider the third equation.

Lemma 4.2. *Let $N \leqslant 4$. Moreover, assume that $C_0 \in L^2(\Omega)$, $\boldsymbol{u} \in W_S$, $T \in Y_S$ and $f_3 \in X_S$. Then the following problem* (29) *has a unique global solution $C \in Y_S$.*

$$\text{(29)} \qquad \begin{cases} \partial_t C - \Delta C + \boldsymbol{u}\cdot\nabla C = \rho\Delta T + f_3 \quad \textit{in} \quad \Omega\times[0, S],\\ \left.\dfrac{\partial C}{\partial n}\right|_{\partial\Omega} = 0, \quad C(\,\cdot\,, 0) = C_0(\,\cdot\,).\end{cases}$$

This problem is quite similar to the previous problem (4). However, we can not use our argument in the proof of Lemma 4.1 directly, since it is not known whether $\Delta T \in X_S$. Therefore, we need some additional argument.

Proof. Let $\chi_\varepsilon \colon [0, S] \to \mathbb{R}$ be the cut-off function defined by

$$\text{(30)} \qquad \chi_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad 0 \leqslant t < \varepsilon,\\ 1 & \text{if} \quad \varepsilon \leqslant t \leqslant S.\end{cases}$$

Since $T \in Y_S$ implies that $\rho\chi_\varepsilon\Delta T \in X_S$, we can show that the following problems have a unique global solution $C_\varepsilon \in Y_S$ for each $\varepsilon > 0$ by applying Lemma 4.1.

$$\text{(31)} \qquad \begin{cases} \partial_t C_\varepsilon - \Delta C_\varepsilon + \boldsymbol{u}\cdot\nabla C_\varepsilon = \rho\chi_\varepsilon\Delta T + f_3 \quad \text{in} \quad \Omega\times[0, S],\\ \left.\dfrac{\partial C_\varepsilon}{\partial n}\right|_{\partial\Omega} = 0, \quad C_\varepsilon(\,\cdot\,, 0) = C_0(\,\cdot\,).\end{cases}$$

Here by showing that $\{C_\varepsilon\}_{\varepsilon>0}$ becomes a Cauchy sequence in Y_S, we can assure the existence of a unique global solution. Indeed, let $\varepsilon_2 > \varepsilon_1 > 0$ and $\delta C = C_{\varepsilon_1} - C_{\varepsilon_2}$.

Moreover, let $\chi_{\tilde{\varepsilon}} = \chi_{\varepsilon_1} - \chi_{\varepsilon_2}$, i.e.,

$$\chi_{\tilde{\varepsilon}}(t) = \begin{cases} 1, & \varepsilon_1 \leqslant t < \varepsilon_2, \\ 0, & \text{otherwise}. \end{cases}$$

Then, δC satisfies the following problem in Y_S:

$$\begin{cases} \partial_t \delta C - \Delta \delta C + \boldsymbol{u} \cdot \nabla \delta C = \rho \chi_{\tilde{\varepsilon}} \Delta T \quad \text{in} \quad [0, S] \times \Omega, \\ \left. \dfrac{\partial \delta C}{\partial n} \right|_{\partial \Omega} = 0, \quad \delta C(\,\cdot\,, 0) = 0. \end{cases} \tag{32}$$

Multiplying (32) by δC, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta C\|_{L^2}^2 + \|\nabla \delta C\|_{L^2}^2 &\leqslant \rho \chi_{\tilde{\varepsilon}} \|\nabla \delta C\|_{L^2} \|\nabla T\|_{L^2} \\ &\leqslant \frac{1}{2}\|\nabla \delta C\|_{L^2}^2 + \frac{\rho^2}{2}\chi_{\tilde{\varepsilon}}^2 \|\nabla T\|_{L^2}^2. \end{aligned} \tag{33}$$

Therefore, integrating (33) over $[0, t]$ and $[0, S]$, we obtain

$$\sup_{0 \leqslant t \leqslant S} \|\delta C(t)\|_{L^2}^2 + \|\nabla \delta C\|_{L^2(0,S;L^2(\Omega))}^2 \leqslant \rho^2 \int_{\varepsilon_1}^{\varepsilon_2} \|\nabla T\|_{L^2}^2 \, ds. \tag{34}$$

Next, multiplying (32) by $-t\Delta \delta C$ and using (24), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} t\|\nabla \delta C\|_{L^2}^2 + \frac{t}{8}\|\Delta \delta C\|_{L^2}^2 \\ &\leqslant \gamma_2 \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 t\|\nabla \delta C\|_{L^2}^2 + \frac{t\rho^2 \chi_{\tilde{\varepsilon}}^2}{2}\|\Delta T\|_{L^2}^2 + \frac{t}{2}\|\delta C\|_{L^2}^2 + \frac{1}{2}\|\nabla \delta C\|_{L^2}^2, \end{aligned} \tag{35}$$

where γ_2 is a general constant independent of ε_1, ε_2. Applying Gronwall's inequality to (35), we obtain

$$\begin{aligned} t\|\nabla \delta C(t)\|_{L^2}^2 \leqslant \int_0^S &\{s\rho^2 \chi_{\tilde{\varepsilon}}^2 \|\Delta T\|_{L^2}^2 + s\|\delta C\|_{L^2}^2 + \|\nabla \delta C\|_{L^2}^2\} \, ds \\ &\times \exp\left(2\gamma_2 \int_0^S \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 \, ds\right). \end{aligned} \tag{36}$$

Therefore, owing to (34), we derive $\sup_{0 \leqslant t \leqslant S} t\|\nabla \delta C(t)\|_{L^2}^2 \leqslant \gamma_2 \int_{\varepsilon_1}^{\varepsilon_2} \{s\|\Delta T\|_{L^2}^2 + \|\nabla T\|_{L^2}^2\} \, ds$ from (36). Moreover, integrating (35) over $[0, S]$, we can get

$$\int_0^S t\|\Delta \delta C\|_{L^2}^2 \, dt \leqslant \gamma_2 \int_{\varepsilon_1}^{\varepsilon_2} \{s\|\Delta T\|_{L^2}^2 + \|\nabla T\|_{L^2}^2\} \, ds. \tag{37}$$

Similarly, multiplying (32) by $t\partial_t\delta C$, we can obtain

$$\int_0^S t\|\partial_t\delta C\|_{L^2}^2\,dt \leqslant \gamma_2\int_{\varepsilon_1}^{\varepsilon_2}\{s\|\Delta T\|_{L^2}^2+\|\nabla T\|_{L^2}^2\}\,ds. \tag{38}$$

Thus, we can assure that $\{C_\varepsilon\}_{\varepsilon>0}$ forms a Cauchy sequence in Y_S since $T\in Y_S$. Hence, the problem (29) has a unique global solution. □

Hence it follows that we can obtain a unique global solution $\underline{T}$, $\underline{C}$ of (2) and the well-definedness of Φ_{T_0,C_0}.

4.2. Well-definedness of Ψ_{u_0}. It is easy to see that (3) has a unique global solution. Indeed, by using the standard arguments of evolution equations governed by maximal monotone operators, we can derive the following (see, e.g., Brézis [3]):

Lemma 4.3. *Let $N\in\mathbb{N}$ and assume that $\boldsymbol{u}_0\in\mathbb{H}_\sigma^1(\Omega)$ and $\boldsymbol{F}\in L^2(0,S;\mathbb{L}_\sigma^2(\Omega))$. Then the following problem* (39) *has a unique global solution $\boldsymbol{u}\in W_S$.*

$$\begin{cases}\partial_t\boldsymbol{u}+\nu\mathcal{A}\boldsymbol{u}+a\boldsymbol{u}=\boldsymbol{F} \quad \text{in} \quad \Omega\times[0,S],\\ \boldsymbol{u}|_{\partial\Omega}=0, \quad \boldsymbol{u}(\,\cdot\,,0)=\boldsymbol{u}_0(\,\cdot\,).\end{cases} \tag{39}$$

Substituting $\boldsymbol{F}$ by $\mathcal{P}_\Omega\boldsymbol{g}\underline{T}+\mathcal{P}_\Omega\boldsymbol{h}\underline{C}+\mathcal{P}_\Omega\boldsymbol{f}_1$, we can assure the existence of a unique global solution $\bar{\boldsymbol{u}}$ of (3) and the well-definedness of $\Psi_{\boldsymbol{u}_0}$.

5. Application of contraction mapping principle

In this section, we assure the local existence of a unique solution of (DCBF) by using Banach's contraction mapping principle.

Let $\underline{\boldsymbol{u}}_i\in W_S$ $(i=1,2)$, $(\underline{T}_i,\underline{C}_i)^t:=\Phi_{T_0,C_0}(\underline{\boldsymbol{u}}_i)$ and $\bar{\boldsymbol{u}}_i:=\Psi_{\boldsymbol{u}_0}((\underline{T}_i,\underline{C}_i)^t)$. Moreover, let $\delta\underline{\boldsymbol{u}}=\underline{\boldsymbol{u}}_1-\underline{\boldsymbol{u}}_2$, $\delta T=\underline{T}_1-\underline{T}_2$, $\delta C=\underline{C}_1-\underline{C}_2$ and $\delta\bar{\boldsymbol{u}}=\bar{\boldsymbol{u}}_1-\bar{\boldsymbol{u}}_2$. Then $\delta\underline{\boldsymbol{u}}$, δT, δC and $\delta\bar{\boldsymbol{u}}$ satisfy the following equations:

$$\begin{cases}\partial_t\delta\bar{\boldsymbol{u}}+\nu\mathcal{A}\delta\bar{\boldsymbol{u}}+a\delta\bar{\boldsymbol{u}}=\mathcal{P}_\Omega\boldsymbol{g}\delta T+\mathcal{P}_\Omega\boldsymbol{h}\delta C,\\ \partial_t\delta T-\Delta\delta T+\underline{\boldsymbol{u}}_1\cdot\nabla\delta T+\delta\underline{\boldsymbol{u}}\cdot\nabla\underline{T}_2=0,\\ \partial_t\delta C-\Delta\delta C+\underline{\boldsymbol{u}}_1\cdot\nabla\delta C+\delta\underline{\boldsymbol{u}}\cdot\nabla\underline{C}_2=\rho\Delta\delta T,\\ \delta\bar{\boldsymbol{u}}|_{\partial\Omega}=0, \quad \left.\dfrac{\partial\delta T}{\partial n}\right|_{\partial\Omega}=0, \quad \left.\dfrac{\partial\delta C}{\partial n}\right|_{\partial\Omega}=0,\\ \delta\bar{\boldsymbol{u}}(\,\cdot\,,0)=0, \quad \delta T(\,\cdot\,,0)=0, \quad \delta C(\,\cdot\,,0)=0.\end{cases} \tag{40}$$

Multiplying the first equation of (40) by $\delta\bar{\boldsymbol{u}}$ and $\mathcal{A}\delta\bar{\boldsymbol{u}}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}^2+\nu\|\nabla\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}^2+a\|\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}^2\leqslant|\boldsymbol{g}|\|\delta T\|_{L^2}\|\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}+|\boldsymbol{h}|\|\delta C\|_{L^2}\|\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2},\\ &\frac{1}{2}\frac{d}{dt}\|\nabla\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}^2+\frac{\nu}{4}\|\mathcal{A}\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}^2+a\|\nabla\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}^2\leqslant\frac{|\boldsymbol{g}|^2}{2\nu}\|\delta T\|_{L^2}^2+\frac{|\boldsymbol{h}|^2}{\nu}\|\delta C\|_{L^2}^2.\end{aligned} \tag{41}$$

Integrating (41) over $[0, t]$ and $[0, S]$, we can derive

$$\begin{aligned}\sup_{0\leqslant t\leqslant S}\|\delta\bar{\boldsymbol{u}}(t)\|_{\mathbb{L}^2} &\leqslant |\boldsymbol{g}|\int_0^S\|\delta T\|_{L^2}\,ds + |\boldsymbol{h}|\int_0^S\|\delta C\|_{L^2}\,ds\\ &\leqslant |\boldsymbol{g}|S\sup_{0\leqslant t\leqslant S}\|\delta T(t)\|_{L^2} + |\boldsymbol{h}|S\sup_{0\leqslant t\leqslant S}\|\delta C(t)\|_{L^2},\end{aligned}\tag{42}$$

and

$$\begin{aligned}&\sup_{0\leqslant t\leqslant S}\|\nabla\delta\bar{\boldsymbol{u}}(t)\|_{\mathbb{L}^2}^2 + \frac{\nu}{2}\int_0^S\|\mathcal{A}\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}^2\,ds\\ &\qquad\leqslant \frac{|\boldsymbol{g}|^2 S}{\nu}\sup_{0\leqslant t\leqslant S}\|\delta T(t)\|_{L^2}^2 + \frac{2|\boldsymbol{h}|^2 S}{\nu}\sup_{0\leqslant t\leqslant S}\|\delta C(t)\|_{L^2}^2.\end{aligned}\tag{43}$$

Next, from the facts that $\int_\Omega \delta T\,\delta\underline{\boldsymbol{u}}\cdot\nabla\underline{T}_2\,dx = -\int_\Omega \underline{T}_2\delta\underline{\boldsymbol{u}}\cdot\nabla\delta T\,dx$ and $\int_\Omega \delta T\,\underline{\boldsymbol{u}}_1\cdot\nabla\delta T\,dx = 0$, multiplying the second equation of (40) by δT, we have

$$\frac{d}{dt}\|\delta T\|_{L^2}^2 + \|\nabla\delta T\|_{L^2}^2 \leqslant \|\underline{T}_2\delta\underline{\boldsymbol{u}}\|_{\mathbb{L}^2}^2 \leqslant \|\underline{T}_2\|_{L^4}^2\|\delta\underline{\boldsymbol{u}}\|_{\mathbb{L}^4}^2 \leqslant \kappa\|\underline{T}_2\|_{H^1}^2\|\delta\underline{\boldsymbol{u}}\|_{\mathbb{H}^1}^2,\tag{44}$$

where κ is a constant depending only on Sobolev's embedding constant. Similarly, multiplying the third equation of (40) by δC, we have

$$\frac{d}{dt}\|\delta C\|_{L^2}^2 + \frac{1}{2}\|\nabla\delta C\|_{L^2}^2 \leqslant \rho^2\|\nabla\delta T\|_{L^2}^2 + 2\kappa\|\underline{C}_2\|_{H^1}^2\|\delta\underline{\boldsymbol{u}}\|_{\mathbb{H}^1}^2.\tag{45}$$

Therefore, we obtain

$$\begin{aligned}&\sup_{0\leqslant t\leqslant S}\|\delta T(t)\|_{L^2}^2 + \int_0^S\|\nabla\delta T\|_{L^2}^2\,dt \leqslant \kappa\sup_{0\leqslant t\leqslant S}\|\delta\underline{\boldsymbol{u}}(t)\|_{\mathbb{H}^1}^2\int_0^S\|\underline{T}_2\|_{H^1}^2\,ds,\\ &\sup_{0\leqslant t\leqslant S}\|\delta C(t)\|_{L^2}^2 \leqslant \kappa\sup_{0\leqslant t\leqslant S}\|\delta\underline{\boldsymbol{u}}(t)\|_{\mathbb{H}^1}^2\left\{\rho^2\int_0^S\|\underline{T}_2\|_{H^1}^2\,ds + 2\int_0^S\|\underline{C}_2\|_{H^1}^2\,ds\right\}.\end{aligned}\tag{46}$$

Hence, combining (42) and (43) with (46), we can derive

$$\begin{aligned}&\sup_{0\leqslant t\leqslant S}\|\delta\bar{\boldsymbol{u}}(t)\|_{\mathbb{H}^1}^2 + \int_0^S\|\mathcal{A}\delta\bar{\boldsymbol{u}}\|_{\mathbb{L}^2}^2\,ds\\ &\leqslant \gamma_3 S(1+S)\sup_{0\leqslant t\leqslant S}\|\delta\underline{\boldsymbol{u}}(t)\|_{\mathbb{H}^1}^2\left\{\int_0^S\|\underline{T}_2\|_{H^1}^2\,ds + \int_0^S\|\underline{C}_2\|_{H^1}^2\,ds\right\},\end{aligned}\tag{47}$$

where γ_3 is a constant depending only on ν, $|\boldsymbol{g}|$, $|\boldsymbol{h}|$, ρ and κ. Since $\int_0^S\|\underline{T}_2(s)\|_{H^1}^2\,ds$ and $\int_0^S\|\underline{C}_2(s)\|_{H^1}^2\,ds$ are bounded by some constant depending only on the initial data and the external forces (see next section), we can assure that $\Psi_{\boldsymbol{u}_0}\circ\Phi_{T_0,C_0}$ becomes a contraction mapping in W_{S_0} with sufficiently small $S_0\in(0, S]$. Namely, we can assure the existence of a unique local solution of (DCBF).

6. Global existence

In this section, we shall show that the unique time-local solution constructed in the previous section can be extended up to S by establishing some a priori estimates.

Multiplying the second equation of (1) by T, we can obtain the following a priori bounds of T for any $S_0 \in (0, S]$ (by using the same procedures as those for (14) and (15)):

$$\sup_{0\leqslant t\leqslant S_0} \|T(t)\|_{L^2}^2 + \|\nabla T\|_{L^2(0,S_0;L^2(\Omega))}^2 \leqslant \gamma_4, \tag{48}$$

where γ_4 is a suitable constant which depends only on $\|T_0\|_{L^2}$ and $\|f_2\|_{L^1(0,S;L^2(\Omega))}$.

Multiplying the third equation of (1) by C, we have

$$\frac{1}{2}\frac{d}{dt}\|C\|_{L^2}^2 + \frac{1}{2}\|\nabla C\|_{L^2}^2 \leqslant \frac{\rho^2}{2}\|\nabla T\|_{L^2}^2 + \|f_3\|_{L^2}\|C\|_{L^2},$$

i.e.,

$$\frac{1}{2}\|C(t)\|_{L^2}^2 \leqslant \frac{1}{2}\gamma_5 + \int_0^t \|f_3(s)\|_{L^2}\|C(s)\|_{L^2}\,ds,$$

where γ_5 denotes some general constant which depends only on ρ, $\|C_0\|_{L^2}$ and γ_4. Here we recall the following variant of Gronwall's inequality (see, e.g., Lemma A.5 of Brézis [3]): if

$$\frac{1}{2}\phi^2(t) \leqslant \frac{1}{2}b^2 + \int_0^t R(s)\phi(s)\,ds$$

is satisfied by $\phi \in C([0,S];\mathbb{R}^1)$, a non-negative constant b and a non-negative $L^1(0,S;\mathbb{R}^1)$-function R, then

$$|\phi(t)| \leqslant b + \int_0^t R(s)\,ds$$

is valid for any $t \in [0, S]$. Hence we easily get

$$\sup_{0\leqslant t\leqslant S_0} \|C(t)\|_{L^2} \leqslant \gamma_6, \tag{49}$$

with some suitable general constant γ_6 which depends only on $\|f_3\|_{L^1(0,S;L^2(\Omega))}$ and γ_5. By the same way as for (41), (42) and (43), multiplying the first equation of (1) by $\boldsymbol{u}$ and $\mathcal{A}\boldsymbol{u}$, we obtain

$$\sup_{0\leqslant t\leqslant S_0} \|\boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \sup_{0\leqslant t\leqslant S_0} \|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 \leqslant \gamma_7, \tag{50}$$

where γ_7 is a constant depending on ν, $\boldsymbol{g}$, $\boldsymbol{h}$, $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$, $\|\boldsymbol{f}_1\|_{L^2(0,S;\mathbb{L}^2(\Omega))}$ and γ_6. Thus we find that there exists a priori bound γ independent of S_0 such that

$$\sup_{0 \leqslant t \leqslant S_0} \{\|T(t)\|_{L^2} + \|C(t)\|_{L^2} + \|\boldsymbol{u}(t)\|_{\mathbb{H}^1_\sigma}\} \leqslant \gamma,$$

which implies that the local solution constructed in previous sections can be extended onto the whole of the prescribed interval $[0, S]$, whence follows our result.

REMARK 3. Throughout our argument in this paper, the positivity of a is not used. Therefore, the existence of a unique solution of (DCBF) still holds even if $a = 0$. We note that for this case, the operator $\boldsymbol{u} \mapsto \mathcal{A}\boldsymbol{u} + a\boldsymbol{u}$ may lose its coercivity in $\mathbb{L}^2_\sigma(\Omega)$.

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Mitsuharu Ôtani
Department of Applied Physics
School of Science and Engineering
Waseda University
3-4-1, Okubo Shinjuku-ku, Tokyo, 169-8555
Japan
e-mail: otani@waseda.jp

Shun Uchida
Department of Applied Physics
School of Advanced Science and Engineering
Waseda University
3-4-1, Okubo Shinjuku-ku, Tokyo, 169-8555
Japan
e-mail: shunuchida@aoni.waseda.jp